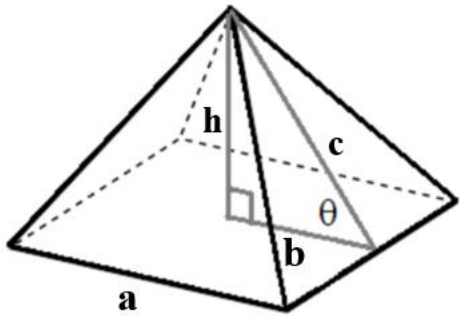
SL Paper 1 Mock A 2020 - WORKED SOLUTIONS v3

Section A

1. The volume of the pyramid is given by



$$V = \frac{1}{3}Ah \Rightarrow 36\sqrt{3} = \frac{1}{3}36h \Rightarrow h = 3\sqrt{3}$$

$$b = \frac{a}{2} = \frac{\sqrt{36}}{2} = 3$$

$$c = \sqrt{h^2 + b^2} = \sqrt{\left(3\sqrt{3}\right)^2 + 3^2} = \sqrt{27+9} = \sqrt{36} = 6$$

$$\Rightarrow \sin\theta = \frac{h}{c} \Rightarrow \theta = \sin^{-1}\left(\frac{h}{c}\right) = \sin^{-1}\left(\frac{3\sqrt{3}}{6}\right) = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

Hence, $\theta = 60°$ $\left[\text{or } \theta = \frac{\pi}{3}\right]$.

2. $$\mathrm{P}(A|B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)} \Rightarrow \mathrm{P}(B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(A|B)} = \frac{1}{5}\cdot\frac{10}{3} = \frac{2}{3}$$

$$\mathrm{P}(A) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B|A)} = \frac{\frac{1}{5}}{\frac{1}{2}} = \frac{2}{5}$$

$$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$$

$$= \frac{2}{5} + \frac{2}{3} - \frac{1}{5} = \frac{13}{15}$$

Hence, $\mathrm{P}(A \cup B) = \frac{13}{15}$

3. (a) $m = 10b + a$

(b) $n - m = 10a + b - (10b + a) = 9a - 9b$

$$\Rightarrow \frac{n-m}{9} = a - b \in \mathbb{Z}$$ since a and b are integers

Therefore, $n - m$ is divisible by 9.

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4. $h'(x) = x\sqrt{1-x^2} = x(1-x^2)^{\frac{1}{2}} \Rightarrow h(x) = \int x(1-x^2)^{\frac{1}{2}}\,dx$

Let $u = 1-x^2 \Rightarrow du = -2x\,dx \Rightarrow -\frac{1}{2}du = x\,dx$

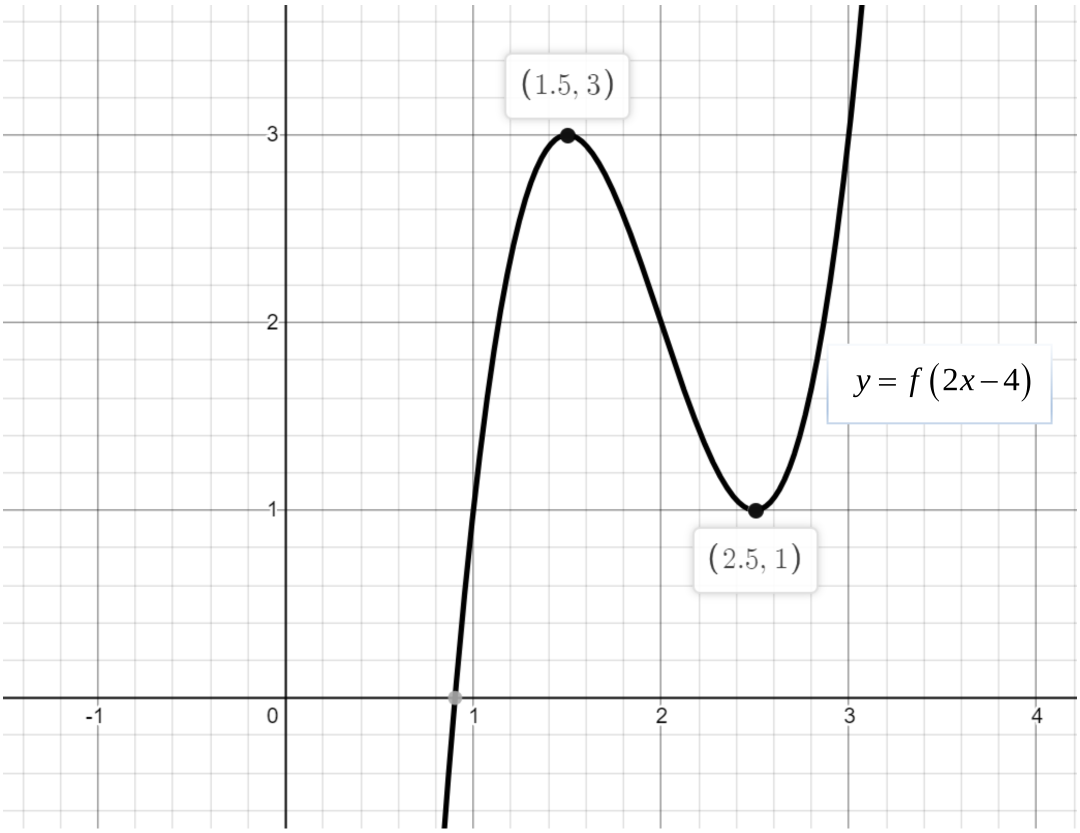
Then,

$$h(x) = -\int \frac{u^{\frac{1}{2}}}{2}du = -\frac{u^{\frac{3}{2}}}{3} + C = -\frac{(1-x^2)^{\frac{3}{2}}}{3} + C$$

$$h(0) = \frac{2}{3}:\ \ h(0) = -\frac{(1-(0)^2)^{\frac{3}{2}}}{3} + C = \frac{2}{3} \Rightarrow C = \frac{2}{3} + \frac{1}{3} = 1$$

Hence, $h(x) = -\frac{(1-x^2)^{\frac{3}{2}}}{3} + 1$ $\left[\text{or } -\frac{1}{3}\sqrt{(1-x^2)^3} + 1\right]$

5. The graph of $y = f[2(x-2)]$ is formed by a sequence of two transformations of the graph of $f(x)$; a horizontal shrink (with respect to the y axis) by a factor of $\frac{1}{2}$, followed by a horizontal translation 2 units to the right



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6. (a) $\ln x + \ln(x-2) - \ln(x+4) = 0$

$\Rightarrow \ \ln(x(x-2)) - \ln(x+4) = 0 \ \Rightarrow \ \ln(x(x-2)) = \ln(x+4)$

$\Rightarrow x(x-2) = x+4 \ \Rightarrow \ x^2 - 3x - 4 = 0$

$\Rightarrow \ (x-4)(x+1) = 0 \ \Rightarrow \ x = 4, \ x = -1$

Verify solutions in original equation:

When $x = 4$: $\ln(4(4-2)) - \ln(4+4) = \ln(8) - \ln(8) = 0$**;** true.

When $x = -1$, The terms $\ln(x)$ and $\ln(x-2)$ are undefined; false.

Hence, $x = 4$

(b) $\log_3(4x^2 - 5x - 6) = 1 + 2\log_3 x$

$\Rightarrow \ \log_3(4x^2 - 5x - 6) = 1 + \log_3(x^2) = \log_3(3) + \log_3(x^2)$

$\Rightarrow \ \log_3(4x^2 - 5x - 6) = \log_3(3x^2) \ \Rightarrow \ 4x^2 - 5x - 6 = 3x^2$

$\Rightarrow x^2 - 5x - 6 = 0 \ \Rightarrow \ (x-6)(x+1) = 0 \ \Rightarrow \ x = 6, \ x = -1$

Verify solutions in original equation:

When $x = 6$:

$\log_3\left(4(6)^2 - 5(6) - 6\right) = \log_3\left(3(6)^2\right) \ \Rightarrow \ \log_3(108) = \log_3(108)$ True.

When $x = -1$, The term $\ln(x)$ is undefined; false.

Hence, $x = 6$

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Section B

7. (a) $f'(x) = \cos x - \sin x$

(b) (i) At maxima and minima:

$f'(x) = \cos x - \sin x = 0 \Rightarrow \cos x = \sin x$

$\Rightarrow \dfrac{\sin x}{\cos x} = \dfrac{\cos x}{\cos x} \Rightarrow \dfrac{\sin x}{\cos x} = 1 \Rightarrow \tan x = 1 \Rightarrow x = \dfrac{\pi}{4} + n\pi,\ n \in \mathbb{Z}$

i.e. the graph $f(x) = \cos x + \sin x$ has maxima and minima at $x = \dfrac{\pi}{4} + n\pi$

When $x = \dfrac{\pi}{4}$: $f\left(\dfrac{\pi}{4}\right) = \sin\dfrac{\pi}{4} + \cos\dfrac{\pi}{4} = \dfrac{\sqrt{2}}{2} + \dfrac{\sqrt{2}}{2} = \sqrt{2}$

Hence, $\mathrm{A}(p,q) = \mathrm{A}\left(\dfrac{\pi}{4}, \sqrt{2}\right)$

(ii) $f''(x) = -\sin x - \cos x$

If A is a maximum, then:

$f''\left(\dfrac{\pi}{4}\right) = -\sin\dfrac{\pi}{4} - \cos\dfrac{\pi}{4} = -\dfrac{\sqrt{2}}{2} - \dfrac{\sqrt{2}}{2} = -\sqrt{2} < 0$

Hence, A is a maximum.

(c) At B, $x = \dfrac{\pi}{4} + \pi = \dfrac{5\pi}{4}$

When $x = \dfrac{5\pi}{4}$: $y = \cos\dfrac{5\pi}{4} + \sin\dfrac{5\pi}{4} = -\dfrac{\sqrt{2}}{2} - \dfrac{\sqrt{2}}{2} = -\sqrt{2}$

Hence, $\mathrm{B}(x,y) = \mathrm{B}\left(\dfrac{5\pi}{4}, -\sqrt{2}\right)$

(d) $r = \sqrt{2},\ c = \dfrac{\pi}{4}$

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8. (a) $\text{P}(\text{1st red ball}) = \frac{4}{6}$, $\text{P}(\text{2nd red ball}) = \frac{3}{5} \Rightarrow \text{P}(\text{2 red balls}) = \frac{4}{6} \cdot \frac{3}{5} = \frac{12}{30} = \frac{2}{5}$

$\text{P}(\text{1st yellow ball}) = \frac{2}{6}$, $\text{P}(\text{2nd yellow ball}) = \frac{1}{5} \Rightarrow \text{P}(\text{2 yellow balls}) = \frac{2}{6} \cdot \frac{1}{5} = \frac{2}{30} = \frac{1}{15}$

$\text{P}(\text{RY}) = \frac{4}{6} \cdot \frac{2}{5} = \frac{4}{15}$, $\text{P}(\text{YR}) = \frac{2}{6} \cdot \frac{4}{5} = \frac{4}{15}$

$\Rightarrow \text{P}(\text{1 red and 1 yellow}) = \text{P}(\text{RY}) + \text{P}(\text{YR}) = \frac{4}{15} + \frac{4}{15} = \frac{8}{15}$

(b) $\text{P}(\text{2 red} \cap \text{A}) = \text{P}(\text{2 red})\text{P}(\text{A}) = \frac{1}{10} \cdot \frac{2}{6} = \frac{1}{30}$

$\text{P}(\text{2 red} \cap \text{B}) = \text{P}(\text{2 red})\text{P}(\text{B}) = \frac{2}{5} \cdot \frac{4}{6} = \frac{4}{15}$

$\Rightarrow \text{P}(\text{2 red}) = \text{P}((\text{2 red} \cap \text{A}) \cup (\text{2 red} \cap \text{B})) = \frac{1}{30} + \frac{4}{15} = \frac{9}{30} = \frac{3}{10}$

(c) $\text{P}(\text{A} \mid \text{2 red}) = \frac{\text{P}(\text{A} \cap \text{2 red})}{\text{P}(\text{2 red})} = \frac{\frac{1}{30}}{\frac{9}{30}} = \frac{1}{30} \cdot \frac{30}{9} = \frac{1}{9}$

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9. (a) $g(x) = \dfrac{x}{e^{x^2}} = xe^{-x^2}$

$g'(x) = u'v + uv'$ Let $u = x,\ v = e^{-x^2}$, then $u' = 1,\ v' = -2xe^{-x^2}$

$\Rightarrow g'(x) = e^{-x^2} - 2x^2e^{-x^2}$

At maxima and minima:

$g'(x) = e^{-x^2} - 2x^2e^{-x^2} = 0 \Rightarrow e^{-x^2}(1-2x^2) = 0 \Rightarrow 1-2x^2 = 0$ since $e^{-x^2} \neq 0$

$\Rightarrow 2x^2 = 1 \Rightarrow x = \pm\sqrt{\dfrac{1}{2}} = \pm\dfrac{\sqrt{2}}{2}$; Since $x \geq 0,\ x = \dfrac{\sqrt{2}}{2}$

P is a maximum if $g''\left(\dfrac{\sqrt{2}}{2}\right) < 0$ [graph of g concave down]

$g''(x) = -2xe^{-x^2} + 4x^3e^{-x^2} - 4xe^{-x^2} = (4x^3 - 6x)e^{-x^2}$

Since $e^{-x^2} > 0$, check if $4x^3 - 6x < 0$

$4\left(\dfrac{\sqrt{2}}{2}\right)^3 - 6\dfrac{\sqrt{2}}{2} = \sqrt{2} - 3\sqrt{2} = -2\sqrt{2} < 0$

Hence, $P\left(\dfrac{\sqrt{2}}{2}, y\right)$ is the one maximum on g.

(b) $g''(x) = (4x^3 - 6x)e^{-x^2}$

Inflexion points exist where $g''(x) = 0$ or $g''(x)$ is undefined; $g''(x)$ is defined for all x

Since $e^{-x^2} > 0$, if $g''(x) = 0$ then $4x^3 - 6x = 0 \Rightarrow 4x^2 = 6 \Rightarrow x^2 = \dfrac{3}{2} \Rightarrow x = \pm\sqrt{\dfrac{3}{2}}$

Since $x \geq 0,\ x = \sqrt{\dfrac{3}{2}}$

When $x < \sqrt{\dfrac{3}{2}}$, e.g. $x = 1$: $g''(1) = (4(1)^3 - 6(1))e^{-(1)^2} = (4-6)e^{-1} < 0$

When $x > \sqrt{\dfrac{3}{2}}$, e.g. $x = 2$: $g''(2) = (4(2)^3 - 6(2))e^{-(2)^2} = (32-12)e^{-4} > 0$

$g''(x) < 0$ for $x < \sqrt{\dfrac{3}{2}}$, $g''(x) > 0$ for $x > \sqrt{\dfrac{3}{2}}$, hence $g(x)$ has an inflexion point $Q\left(\sqrt{\dfrac{3}{2}}, y\right)$

(c) (i) $x > \sqrt{\dfrac{3}{2}}$ (ii) $0 \leq x < \sqrt{\dfrac{3}{2}}$

(d) $\int_0^k xe^{-x^2}\,dx = \dfrac{1}{2} - \dfrac{1}{2e^4}$ Let $u = e^{-x^2}$, then $\dfrac{du}{dx} = -2xe^{-x^2} \Rightarrow du = -2xe^{-x^2}dx$

$\Rightarrow \int_0^k xe^{-x^2}\,dx = \int_0^k -\dfrac{du}{2} = \left[-\dfrac{u}{2}\right]_0^k = \left[-\dfrac{e^{-x^2}}{2}\right]_0^k = -\dfrac{e^{-k^2}}{2} + \dfrac{1}{2} = \dfrac{1}{2} - \dfrac{1}{2e^{k^2}} = \dfrac{1}{2} - \dfrac{1}{2e^4}$

Hence, $k^2 = 4 \Rightarrow k = 2$